The Klobüršteltheorem

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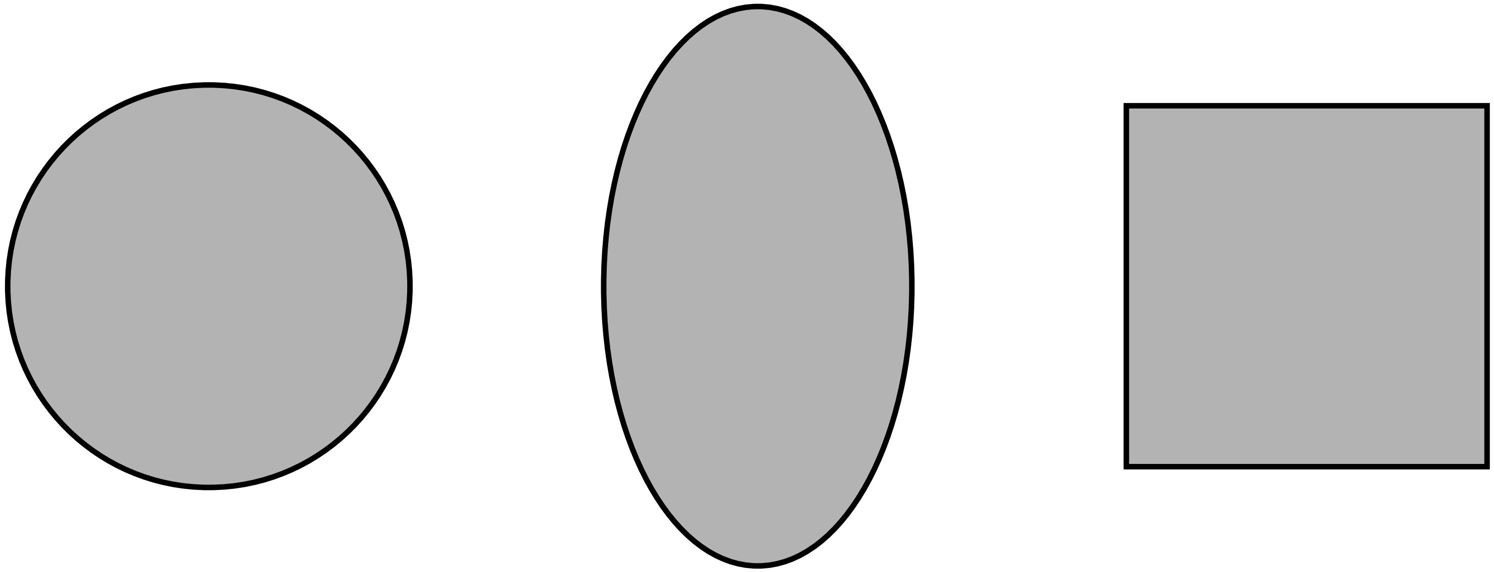
The Faber-Krahn Theorem

Among all bounded domains $D \subset \mathbb{R}^n$ with fixed volume, a ball has the lowest first Dirichlet eigenvalue.

$$-\Delta u = \lambda u \,, \qquad u|_{\partial D} = 0$$



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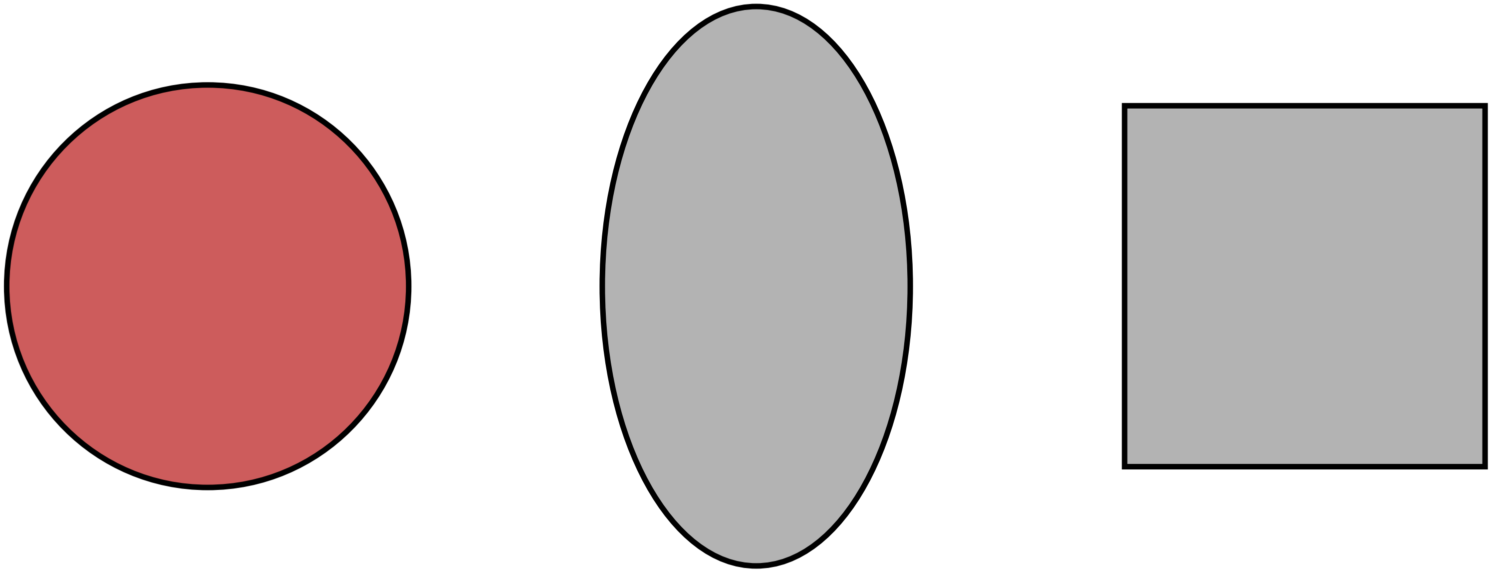

Graph Laplacian

$G = (V, E)$ simple graph with vertex set V, edge set E
(and possibly weights $\frac{1}{c_e} > 0$).

Laplacian of G

$$\Delta = \Delta(G) = D(G) - A(G)$$

$A(G) \ldots$ adjacency matrix.
$D(G) \ldots$ diagonal matrix with vertex degrees

Contrary to the “classical” Laplace-Beltrami operator on manifolds, the graph Laplacian $\Delta(G)$ is defined as a **positive** operator.

Graph with Boundary

A **graph with boundary** is a graph $G(V_0 \cup \partial V, E_0 \cup \partial E)$

V_0 ... interior vertices

∂V ... boundary vertices

E_0 ... edges between interior vertices (interior edges)

∂E ... edges between boundary and interior vertices

(boundary edges)

We assume that all boundary vertices have degree 1
(and vice versa).

Discrete Dirichlet Operator

A **discrete Dirichlet operator** Δ_0 is the graph Laplacian restricted to interior vertices, i.e.

$$\Delta_0 = D_0 - A_0$$

where A_0 is the adjacency matrix of the graph induced by the interior vertices, $G(V_0, E_0)$, and where D_0 is the degree matrix with the vertex degrees in the whole graph $G(V_0 \cup \partial V, E_0 \cup \partial E)$ as its entries.

Faber-Krahn Property

We say that a graph with boundary has the **Faber-Krahn property** if it has lowest first Dirichlet eigenvalue among all graphs with the same “volume” in a particular graph class.

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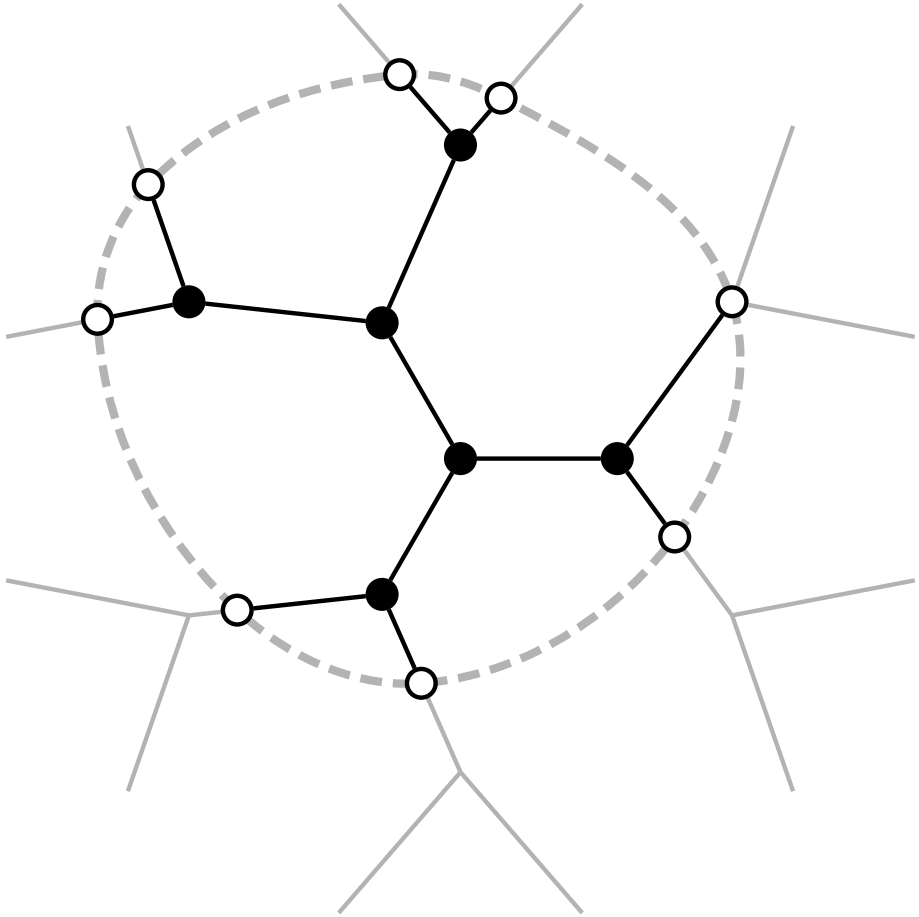
This definition raises two questions:

(1) What is the "volume" of a graph, and

(2) what is an appropriate graph class?

Friedman’s Class

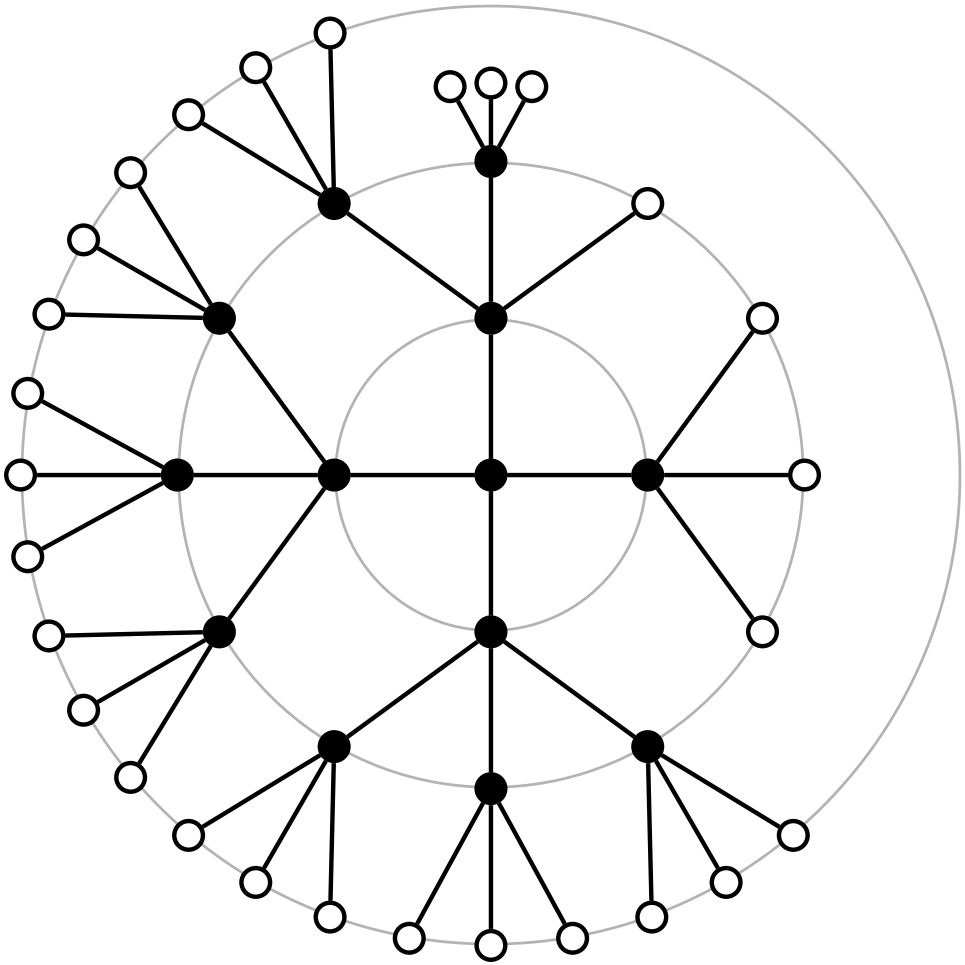


The class of trees that can be obtained by cutting subsets out of the geometric representation of an infinite

d-regular tree.

Volume is total length of edges.

Onion-Shaped Trees



Example of tree with Faber-Krahn property in Friedman's graph class with volume 38.5.

Faber-Krahn Theorem for Friedman's class

Regular trees with the Faber-Krahn property are not balls, but almost balls with a complicated structures
("lazy peeled onions").

Nonregular Trees

When generalizing the Faber-Krahn type theorems to arbitrary trees, the picture of cutting out a graph fails. Instead . . .

Problem
Given a class $\mathcal{C}$ of graphs, where all graphs have the same "volume". Now characterize all graphs in $\mathcal{C}$ with the Faber-Krahn property, i.e., which minimize the first Dirichlet eigenvalue.

It seems natural to use the number of vertices as measure for the "volume" of a graph (equivalent to number of edges).

Analogous results for the Laplace-Beltrami-operators on manifolds with non-constant curvature are rare.

Too Simple: Class of All Trees

Let us consider the class $\mathcal{T}^{(n)}$ of all connected trees with n vertices (and at least two boundary vertices).
The volume is the total number of vertices, i.e. n,

Theorem (Katsuda & Urakawa 1999)
A graph T has the Faber-Krahn property in class $\mathcal{T}^{(n)}$ if and only if it is a path of length $n-1$.

More Appropriate Classes

Class of all trees with number of interior and boundary vertices fixed:

$$\mathcal{T}^{(n,k)} = \{G \text{ is a tree, with } |V| = n \text{ and } |V_0| = k\}$$

Class of all trees with number of interior and boundary vertices fixed with minimum vertex degree:

$$\mathcal{T}_d^{(n,k)} = \{G \in \mathcal{T}^{(n,k)} \colon d_v \geq d \text{ for all } v \in V_0\}$$

Class of all trees with degree sequence fixed:

$$\mathcal{T}_\pi = \{G \text{ is a tree with boundary with degree sequence } \pi\}$$

Degree Sequence

A sequence $\pi = (d_0, \ldots, d_{n-1})$ of nonnegative integers is called **degree sequence** if there exists a graph G with n vertices for which $d_0, \ldots, d_{n-1}$ are the degrees of its vertices.

In the following we assume that the degrees sequence of G is given by $\pi = (d_0, d_1, \ldots, d_{k-1}, d_k, \ldots, d_{n-1})$ such that the degrees d_i are non-decreasing for $0 \leq i < k = |V_=|$, and $d_j = 1$ for $j \geq k$ (i.e., correspond to boundary vertices).

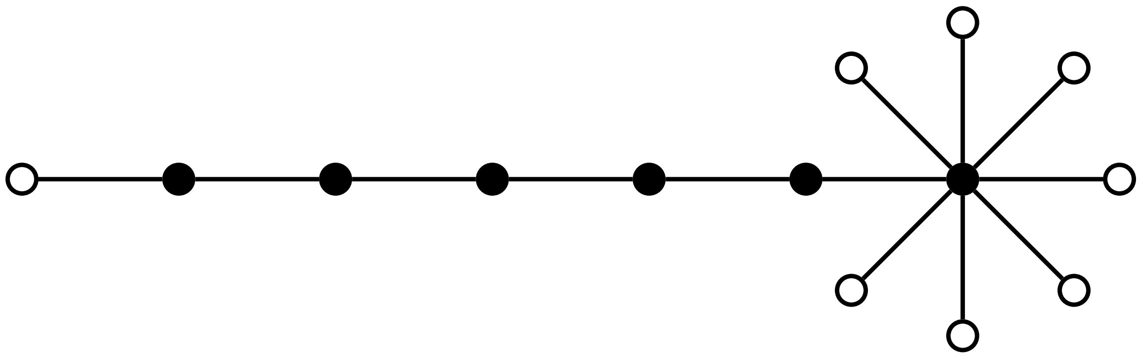
Proposition (Harary 1969)
A degree sequence $\pi = (d_0, \ldots, d_{n-1})$ is a tree sequence (i.e. a degree sequence of some tree) if and only if every $d_i > 0$ and $\sum_{i=0}^{n-1} d_i = 2\,(n-1)$.

The Klobüršteltheorem

Theorem

A tree G has the Faber-Krahn property in a class $\mathcal{T}$ if and only if G is a star with a long tail, i.e. a comet (aka *Klobüršte*l).

G is then uniquely determined up to isomorphism.



Height, Parent, and Child

For a tree G with root v_0 the **height** $h(v)$ of a vertex v is defined by $h(v) = \text{dist}(v, v_0)$.

For two adjacent vertices v and w with $h(w) = h(v) + 1$ we call v the **parent** of w, and w a **child** of v.

Notice that every vertex $v \neq v_0$ has exactly one parent, and every interior vertex w has at least one child vertex.

SLO-Ordering

A well-ordering $\prec$ of the vertices of a tree with boundary is called **spiral-like** if the following holds:

(S1) $v \prec w$ implies $h(v) \leq h(w)$;

(S2) if $v_1 \prec v_2$ then for all children w_1 of v_1 and all children w_2 of v_2, $w_1 \prec w_2$;

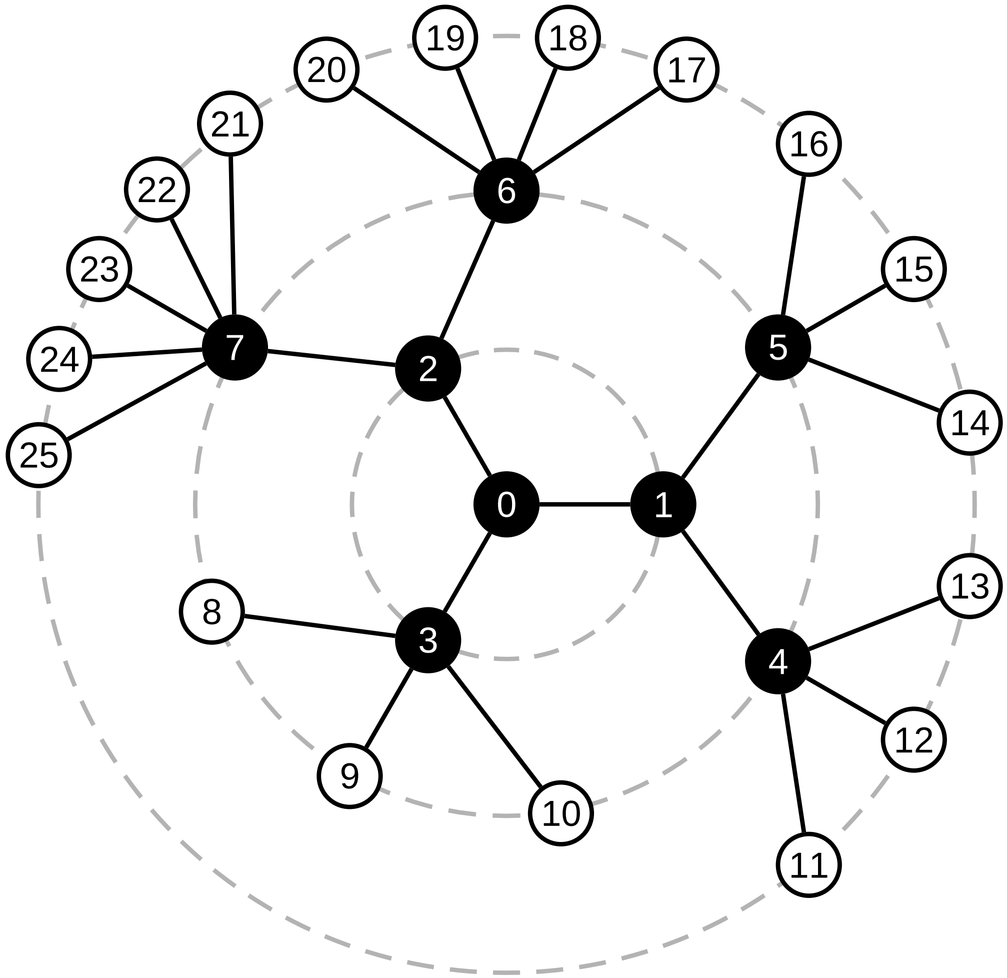
(S3) if $v \prec w$ and $v \in \partial V$, then $w \in \partial V$.

It is called **spiral-like with increasing degrees** (**SLO***-ordering for short) if additonally the following holds

(S4) if $v \prec w$ for interior vertices $v, w \in V_0$, then $d_v \leq d_w$.

We call trees that have a SLO- or SLO*-ordering of its vertices **SLO-trees** and **SLO*-trees**, respectively.

SLO-Ordering



Faber-Krahn Theorems

Theorem 1

A tree G has the Faber-Krahn property in a class $\mathcal{T}$ if and only if G is a star with a long tail, i.e. a comet (aka *Klobürštel*).

Theorem 2

A graph G has the Faber-Krahn property in a class $\mathcal{T}_d$ if and only if it is a SLO*-tree where all but one interior vertices have degree d.

Theorem 3

A graph G with degree sequence π has the Faber-Krahn property in the class $\mathcal{T}_\pi$ if and only if it is a SLO*-tree.

G is then uniquely determined up to isomorphism.

The Rayleigh Quotient

The **Rayleigh quotient** on a real-valued function f is the fraction

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} = \frac{\sum_{(u,v)\in E} (f(u) - f(v))^2}{\sum_{v \in V} f(v)^2}.$$

We denote the first Dirichlet eigenvalue of a graph G by $\lambda(G)$.

Proposition

$$\lambda(G) = \min_{f \in \mathcal{S}} \mathcal{R}_G(f) = \min_{f \in \mathcal{S}} \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle}$$

where $\mathcal{S}$ is the set of all real-valued functions on V with the constraint $f|_{\partial V} = 0$.

Rearrangements

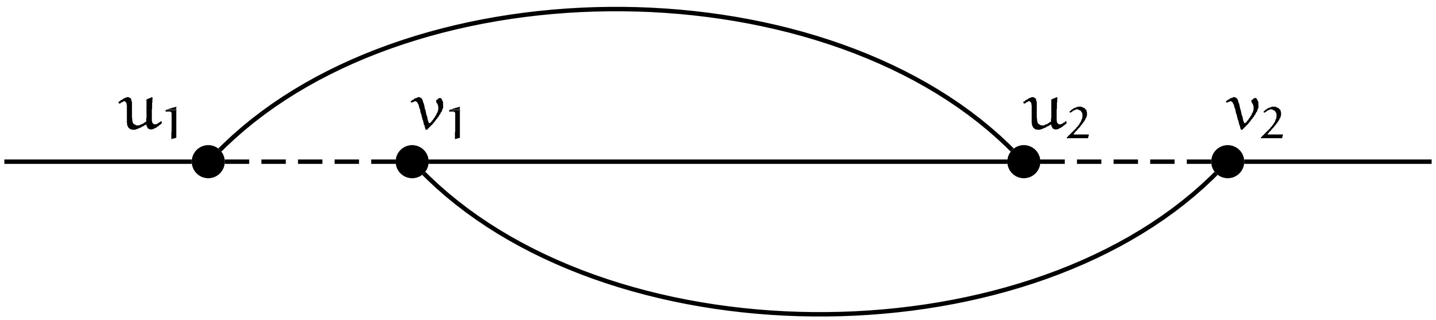
The main techniques for proving our theorems is **rearranging** of edges.

We need two different types of rearrangement steps that we call **switching** and **shifting**, respectively.

Starting with graph $G(V, E)$ we move edges in such a step to get a new graph $G'(V, E')$.

For each step we are able to show that the Rayleigh quotients is non-increasing for a particular function on V.

Switching



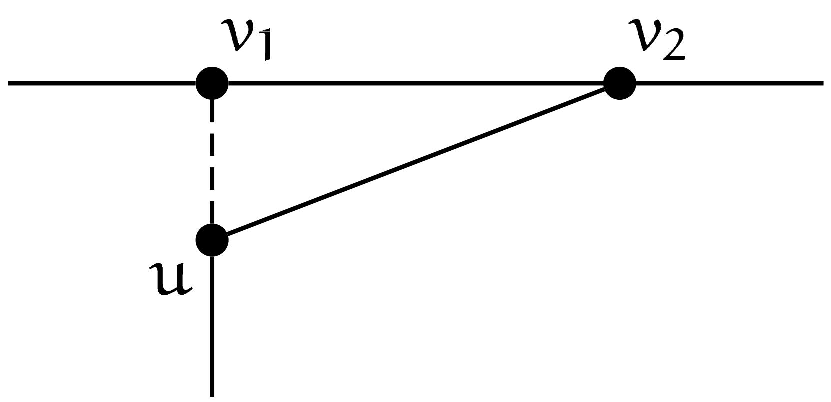
For a non-negative function $f \in \mathcal{S}$ we find

$$\mathcal{R}_{G'}(f) \leq \mathcal{R}_G(f)$$

whenever $f(v_1) \geq f(u_2)$ and $f(v_2) \geq f(u_1)$.
This inequality is strict if both inequalities are strict.

$\mathcal{T}_\pi$ is closed under switching.

Shifting



For a non-negative function $f \in \mathcal{S}$ we find

$$\mathcal{R}_{G'}(f) \leq \mathcal{R}_G(f)$$

if and only if $f(v_1) \geq f(v_2)$.

The inequality is strict if $f(v_1) > f(v_2)$.

$\mathcal{T}_2$ is closed under shifting, but $\mathcal{T}_\pi$ is not!

Proof: A Greedy Algorithm

Input: Tree $G(V, E) \in \mathcal{T}_\pi$ with non-negative eigenfunction f to $\lambda(G)$.
Output: Tree $G^*(V, E^*) \in \mathcal{T}_\pi$ with SLO-ordering $\prec$ and $\lambda(G^*) \leq \lambda(G)$.

1. Enumerate vertices such that $f(v_i) > f(v_j)$ implies $i < j$.
2. Define a well-ordering $\prec: v_i \prec v_j$ if and only if $i < j$.
3. Set $s \leftarrow 0$.
4. For $r = 0, \ldots, k-1$ do
5.   For $i = 1, \ldots, d_r - 1$ do   [$i = 1, \ldots, d_0$ if $r = 0$]
6.   Set $s \leftarrow s + 1$ (increment s).
7.   If v_s is not adjacent to v_r then
8.   Select an edge (v_r, w_r) such that $v_s \prec w_r$.
9.   Select an edge (v_s, w_s) such that $v_s \prec w_s$ and
   w_s is in the geodesic path from v_r to v_s if and only if w_r is not.
10.   Apply Switching such that the new graph G_s has edges (v_r, v_s) and (w_r, w_s).
11. Forall $(v, v_r) \in E$ with $v_s \prec v$ do
12.   Apply Shifting such that edge (v, v_r) is replaced by edge (v, v_{r+1}).

Proof (Cont.)

Hard work: Show that this greedy algorithm

(i) works;

(ii) always results in graphs as described in the above theorems;

(iii) which are isomorphic they belong to the same graph class.

Idea: for each step in this iteration use the above lemmata to show that the Rayleigh quotient is non-increasing.

[Details are tedious and thus skipped . . .]

Further results

Theorem

Let $G(V, E)$ have the Faber-Krahn property in $\mathcal{T}_\pi$
and $G'(V', E')$ have the Faber-Krahn property in $\mathcal{T}_{\pi'}$
for two degree sequences with $|\pi| = |\pi'| = n$ that satisfy
$\sum_{j \le r} d_j \le \sum_{j \le r} d'_j$ for all $0 \le r < n$.

Then $\lambda(G) \le \lambda(G')$, where equality holds if and only if $\pi = \pi'$.