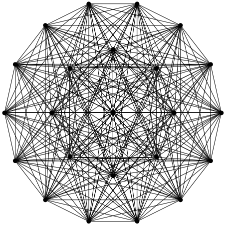

Dilation coefficients of complete graphs

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Repeat: Stochastic proximity embedding (SPE)

The SPE algorithm is using the idea of stochastic proximity embedding (introduced by D. K. Agrafiotis).

We are given relations/proximities/distances between n objects.

SPE for a pre-described number of steps extends or contracts edges; in our case, the algorithm at every step modifies a random edge to be of length one. The algorithm starts with a random representation and iteratively refines it by repeatedly selecting an edge at random and adjusting its coordinates, so that the edge length becomes one. The magnitude of these adjustments is controlled by a temperature, which decreases during the course of the simulation to avoid oscillations.

The algorithm is programmatically simple, robust, convergent and scales linearly with respect to sample size.

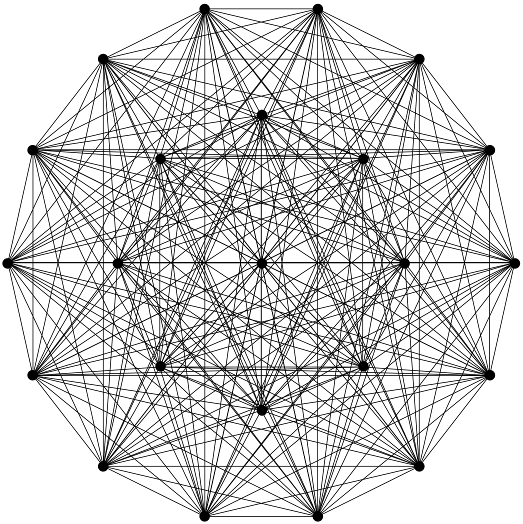
Motivation

W. Andreas Svrcek-Seiler asked at 2004 Winterseminar in his talk 0.02 € on Embedding, the following questions:

*... Without any mathematical rigor one might state that SPE algorithm allows "nice" representations of *some* graphs, especially if they are highly connected.*

- *Why is that?*
- *When using SPE algorithm with all proximities equal to one, why are objects always drawn similar to a representation on the following slide?*
- *How does that some objects are getting mapped inside the representation and not on the boundary?*
- *Why does the number of interior objects differ with n?*

Symmetric concentric representation of K_{23}



Symmetric planar representation of the complete graph K_{23}.

The energy of a graph representation

Usually the algorithms for automatic drawing of graphs are based on the local search method, where the total energy of the drawing is being minimized.

In graph drawing algorithms usually only the distances of adjacent and sometimes also non-adjacent vertices are taken into consideration.

An example: Spring embedders

Spring embedders: Edges are modeled with springs. Two forces are defined (the attractive force acts between adjacent vertices and the repulsive one between all pairs of vertices). The force between two vertices depends on the Euclidean distance between them. The minimal energy corresponds to the equilibrium point of the forces.

The following equation is the energy function for the well-known spring embedding algorithm of type Fruchterman-Reingold:

$$\mathcal{E}_{\mathrm{FR}}(\rho, G) = \sum_{v \in V} \left(\sum_{u \in N(v)} \frac{\|\rho(u) - \rho(v)\|^3}{3k} - k^2 \sum_{u \in V} \log \|\rho(u) - \rho(v)\| \right),$$

where $N(v)$ is the set of all neighbor vertices of vertex v in the graph G, and where $V = V(G)$.

The energy of a graph representation

The "energy" that is being investigated in this talk is simply the quotient of the longest and the shortest edge representation. Such quotient is called the dilation coefficient of the representation.

The minimum of all dilation coefficients over all planar representations of graph G is called the dilation coefficient $\Delta(G)$ of a graph.

It is a graph invariant.

The dilation coefficient

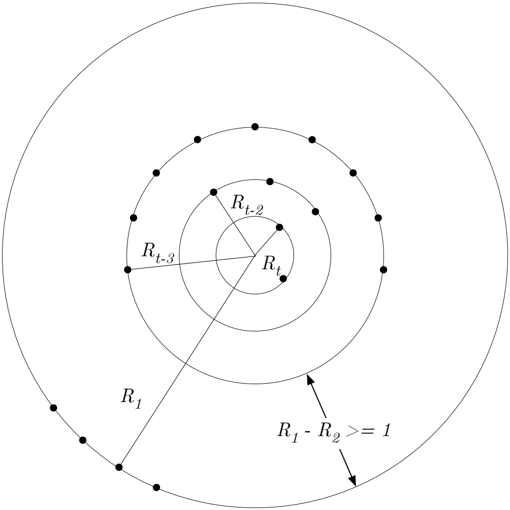
Graphs for which $\Delta(G) = 1$ are quite special as they can be drawn in the plane with all edges of the same length. Such graphs are called unit-distance graphs.

As opposed to unit-distance graphs, the complete graphs have the maximal possible dilation coefficient for a given number of vertices.

In this talk, we will observe $\mathcal{E}_\infty(\rho, G) := \max_{e \in E(G)} \|\rho(e)\|$.

Considering $\min_{e \in E(G)} \|\rho(e)\| = 1$, algorithms that minimize the energy function $\mathcal{E}_\infty$ also minimize the dilation coefficient of G.

The upper bound for Δ - the idea



Every graph representation gives the upper bound for the dilation coefficient. We present the idea of a symmetric concentric representation of n vertices in $\mathbb{R}^2$.

General non-uniform concentric representation

Let $r_m = \frac{1}{2\sin\left(\frac{\pi}{m}\right)}$ be the radius of the circumscribed circle of the regular unit side m-gon.

The idea is to place some vertices onto the outer orbit with the smallest radius as possible and to optimally place all remaining vertices (in a recursive way) inside of the outer orbit. To achieve this, we observe two situations:

1. We try to place $n - m$ vertices into points of a regular $(n - m)$-gon with side one and remaining m vertices inside a disc with the radius smaller than $r_{n-m} - 1$.
2. We try to place $n - m$ vertices into points of a regular $(n - m)$-gon with points circularly embedded onto a circle with radius $R_m + 1$, where R_m is the radius of the circumscribed circle of the smallest disc containing the remaining m vertices.

General non-uniform concentric representation

Define an ordered integer partition $[m_1, m_2, \ldots, m_t]$ of a natural number n, where $n = m_1 + m_2 + \ldots + m_t$ and $m_1 > m_2 > \ldots > m_t > 0$.

The general (or non-uniform) concentric representation $\rho_{[m_1, m_2, \ldots, m_t]}(K_n)$ of the complete graph K_n with respect to the ordered integer partition $[m_1, m_2, \ldots, m_t]$, is defined as follows: *we place n vertices on t concentric cycles, m_i vertices into m_i evenly spaced points on the i-th cycle with radius R_i, such that every pair of such points is at distance at least one and such that two neighboring cycles are at least one apart; i.e.,* $R_i = \max\{r_{m_i}, R_{i+1} + 1\}$.

General non-uniform concentric representation

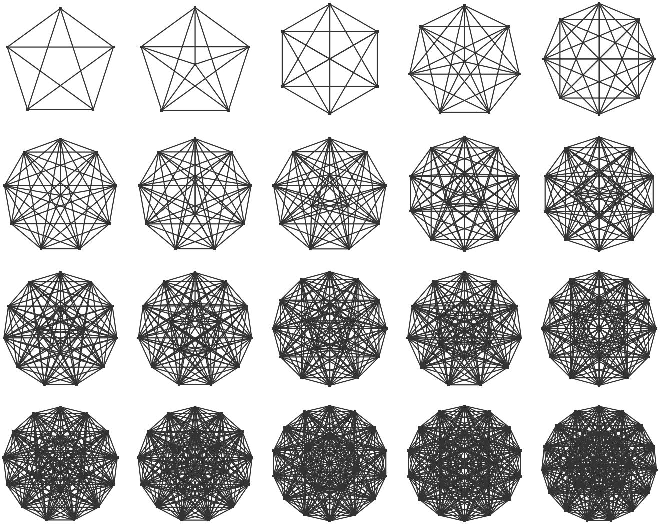
An ordered integer partition $[m_1, m_2, \ldots, m_t]$ of an integer n that gives the optimal Δ could be calculated using a dynamic programming method with the top-down approach and the memorization.

The Bellman equation for recursively calculating the smallest radius R_n of the outer cycle of an optimal general concentric representation of the complete graph K_n, is given by

$$R_n = \min_{0 \leq m < n} \left\{ \{ r_{n-m} \mid r_{n-m} - R_m > 1 \} \bigcup \{ R_m + 1 \mid r_{n-m} - R_m \leq 1 \} \right\}.$$

Concentric radii give the general concentric ordered integer partition, which defines the general concentric representation and its dilation coefficient.

Symmetric concentric representations of $K_5, \ldots, K_{24}$



Symmetric general non-uniform concentric representations of complete graphs K_n, $5 \leq n \leq 24$.

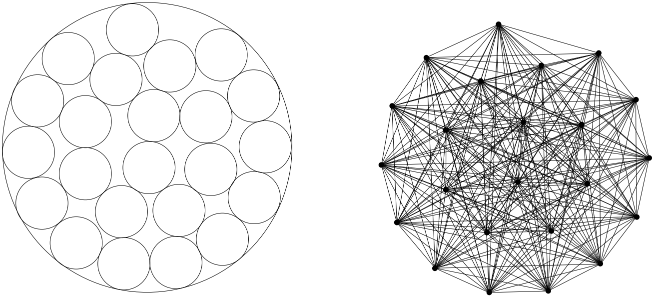
Circular packing

Our problem is, in addition to famous Erdös problems, related to the problem of packing unit circles into a circle with the smallest radius.

Namely, if we represent the vertices of K_n as the centers of the unit circles, every vertex is at least at distance $\frac{1}{2}$ from the boundary of the larger circle and so the maximum distance between any two vertices of K_n is at most $r - 1$, where r is the radius of the larger circle.

The circular packing problem uses only discs with diameter exactly one and hence, places centers of each two discs at distance at least one. This is not necessary when obtaining the dilation coefficient of the complete graph; and hece, the problems are not the same.

Circular packing : An example



(a) Do sedaj znano najboljše pakiranje kroga s 23 enotskimi krogi. (b) Polni graf K_{23} prirejen temu pakiranju ima dilacijski koeficient 4,5445.

Conclusions : Comparisons

		Standard uniform concentric representation	General concentric representation		Circular packing [20]				
n	Lower bound [18,19]	Upper bound $\Delta_{\{m_1,\dots\}}(K_n)$	Upper bound $\Delta_{[m_1,\dots]}(K_n)$	Ordered integer partition $[m_1,\dots]$	Upper bound $\Delta_{CP}(K_n)$	$\frac{\Delta_{CP}(K_n)}{\Delta_{[m_1,\dots]}(K_n)}$	SAAlg	FRAlg	SPEAlg
2	0.4850	2	1	{2}	1	100.00%	1	1	1
3	0.8188	2	1	{3}	1	100.00%	1	1	1
4	1.1002	2	1.4142	{4}	1.4142	100.00%	1.4142	1.4164	1.4201
5	1.3480	2	1.6180	{5}	1.6180	100.00%	1.6297	1.6383	1.6374
6	1.5722	2	1.9021	{5, 1}	2	105.15%	1.9044	2.0781	1.9906
7	1.7782	2	2	{6, 1}	2	100.00%	2.0055	2.0566	2.0795
8	1.9701	4	2.2470	{7, 1}	2.2470	100.00%	2.2568	2.3307	2.3591
9	2.1502	4	2.6131	{8, 1}	2.6131	100.00%	2.6979	2.8162	2.7269
10	2.3206	4	2.8794	{9, 1}	2.7938	97.03%	3.3205	3.3825	3.0530
11	2.4827	4	2.9544	{9, 2}	2.8794	97.46%	3.3523	3.6682	3.2387
12	2.6376	4	3.1068	{9, 3}	2.9960	96.43%	3.1614	3.7134	3.5829
13	2.7861	4	3.2361	{10, 3}	3.2361	100.00%	3.4372	4.0166	3.7671
14	2.9290	4	3.4142	{10, 4}	3.3251	97.39%	3.6075	4.3921	4.0907
15	3.0669	4	3.5133	{11, 4}	3.5202	100.19%	3.9151	4.2499	4.5065
16	3.2003	4	3.6636	{11, 5}	3.5933	98.08%	3.7454	4.6231	4.8885
17	3.3296	4	3.8637	{12, 5}	3.7837	97.93%	4.0499	4.6431	4.6870
18	3.4551	4	3.9593	{11, 6, 1}	3.8637	97.59%	3.9928	5.1883	5.2157
19	3.5772	4	4	{12, 6, 1}	3.8637	96.59%	4.2001	5.1174	5.6607
20	3.6961	6	4.1481	{13, 6, 1}	4.1015	98.88%	4.4755	5.2817	5.9600
21	3.8121	6	4.2734	{13, 7, 1}	4.2348	99.10%	5.2243	5.7495	6.5630
22	3.9253	6	4.4940	{14, 7, 1}	4.4389	98.77%	5.4136	5.6523	6.8137
23	4.0360	6	4.6131	{14, 8, 1}	4.5445	98.51%	5.3458	5.8755	6.8752
24	4.1443	6	4.7834	{15, 8, 1}	4.6449	97.10%	5.6482	5.7917	7.4315
25	4.2504	6	4.8968	{15, 9, 1}	4.7526	97.05%	5.8130	6.4269	7.6314

Upper bounds for the dilation coefficient of the complete graph K_n on n vertices compared to dilation coefficients of the representations obtained by several graph-drawing algorithms.

Further reading

- B. Horvat, T. Pisanski, A. Žitnik, The dilation coefficient of a complete graph, to be published in CCA (2009).
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- M. Kaminski, P. Medvedev and M. Milanič, The plane-width of graphs, submitted (2009).

Thank you!

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