Hypergraph Products

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Bioinformatik Leipzig

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Outline

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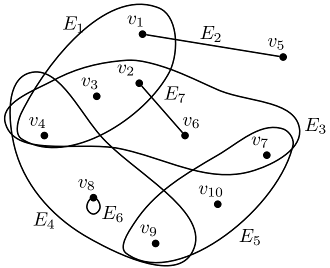
1. Hypergraph Products
   - The Cartesian Product
   - The Direct Product
   - The Strong Product
2. Prime Factorization w.r.t. the Cartesian product

Hypergraphs

- A hypergraph is a pair $H = (V, \mathscr{E})$ with vertex set $V \neq \emptyset$ and a family of edges $\mathscr{E}$ where the edges are subsets of V.



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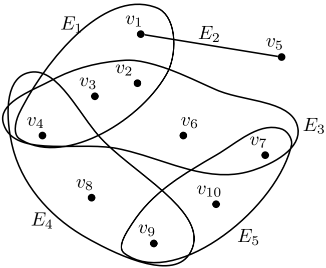

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Hypergraph Products

- For all products $H_1 \star H_2$:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

1. Restriction to graphs are the common graph products
2. Associativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. The projections $p_i : V(H_1 \star H_2) \to V(H_i)$ for $i \in \{1,2\}$ are at least weak homomorphisms
7. Unique prime factorization

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The Cartesian Product

Edge set of the Cartesian Product $H_1 \Box H_2$ of two hypergraphs H_1, H_2

$$\mathscr{E}(H_1 \Box H_2) = \Big\{ \{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2) \Big\} \\ \cup \Big\{ E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2) \Big\}$$

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I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \Box H_2)$

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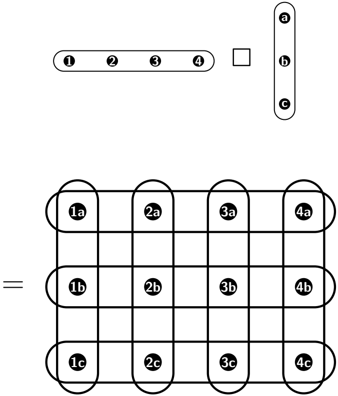

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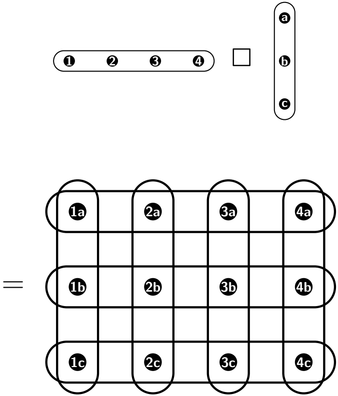

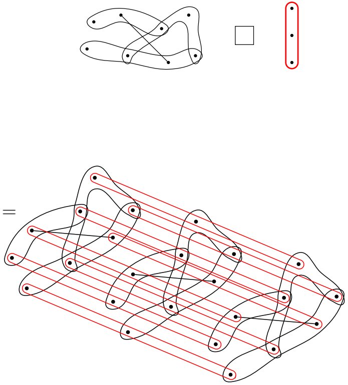

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$$\mathscr{E}(H_1 \times H_2) := \big\{E \subseteq V(H_1) \times V(H_2) \,|\, p_1(E) \in \mathscr{E}(H_1),\ p_2(E) \in \mathscr{E}(H_2) \text{ and } |E| = \max(|p_1(E)|, |p_2(E)|)\big\}$$

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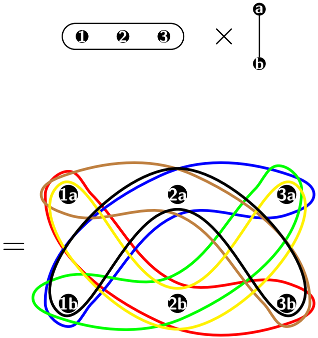

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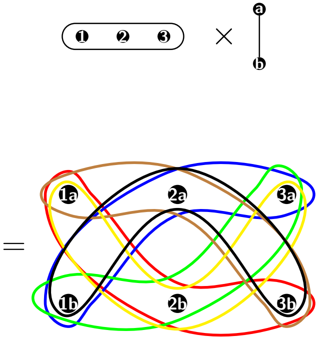

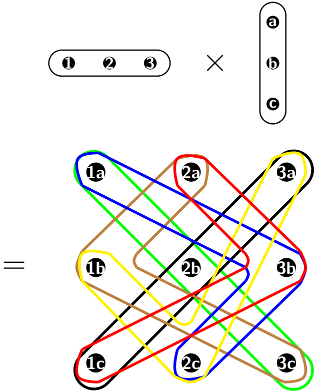

Properties of the direct product:

- Restriction to graphs is the direct graph product
- Associativity
- Commutativity
- Distributivity w.r.t. the disjoint union
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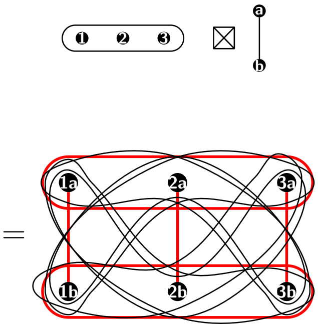

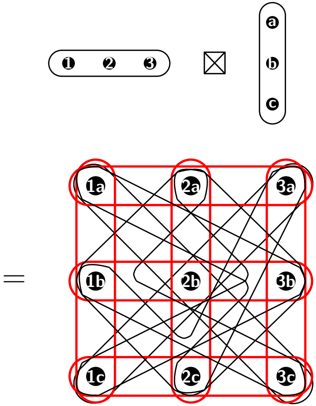

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The strong product is a special case of the direct product:

- $\mathcal{L}H := (V(H), \mathscr{E}(H) \cup \{\{v\} \mid v \in V(H)\}),$
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$$H_1 \boxtimes H_2 = \mathcal{N}(\mathcal{L}H_1 \times \mathcal{L}H_2)$$

for simple hypergraphs H_1, H_2

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Prime Factorization w.r.t the Cartesian Product

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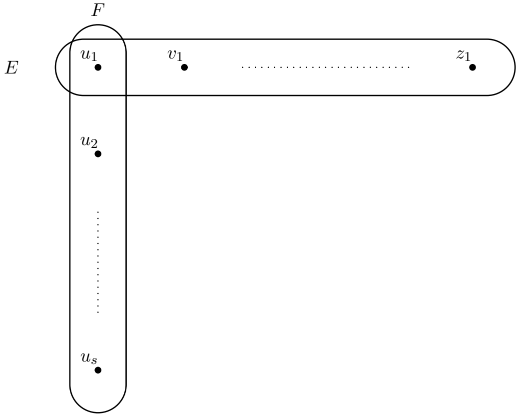

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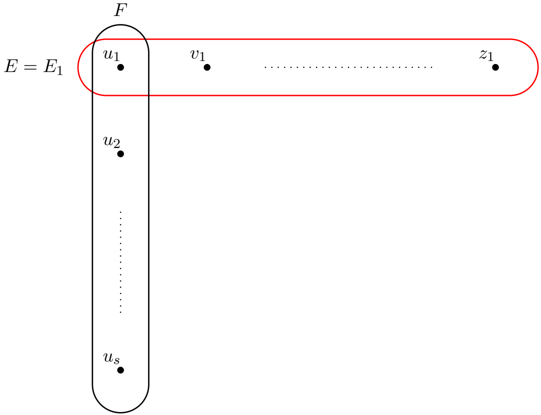

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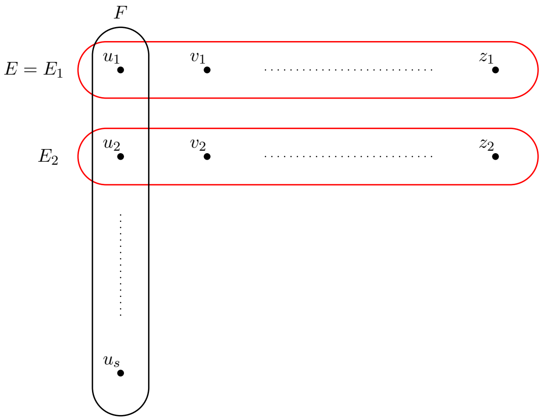

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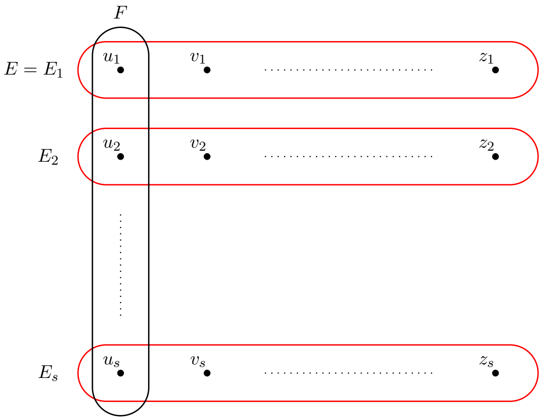

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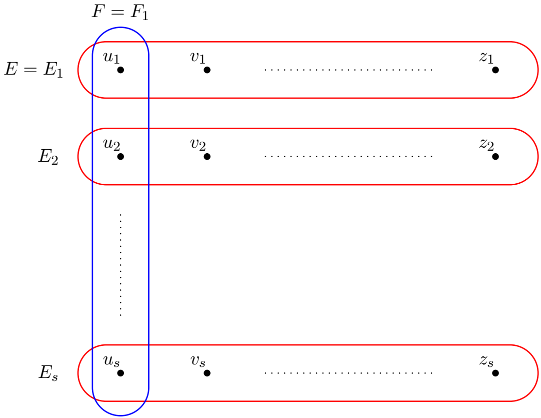

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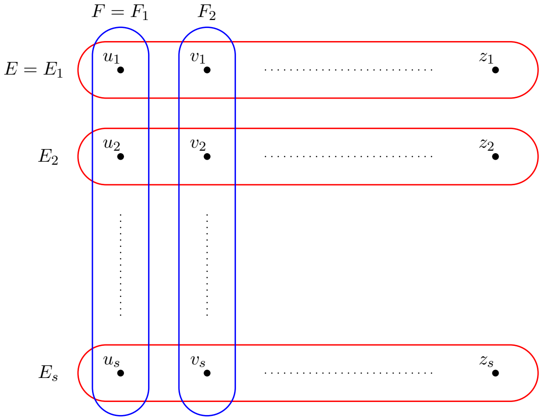

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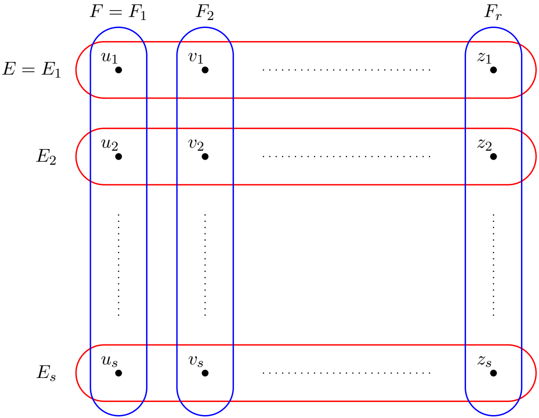

The Relation δ

"'starting"'-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) E and F are opposite edges of a four-cycle

(ii) $E \cap F \neq \emptyset$ and $\nexists(|E| \times |F|)$-grid without diagonals containing them.

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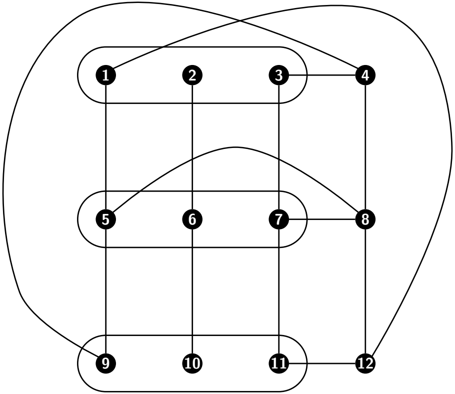

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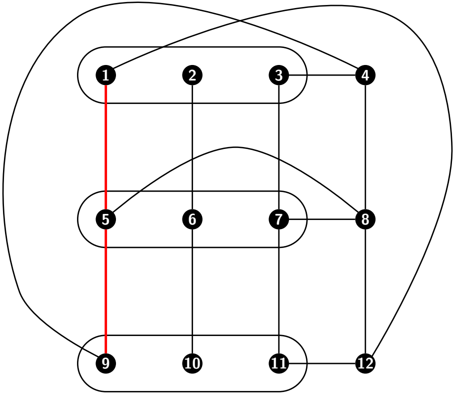

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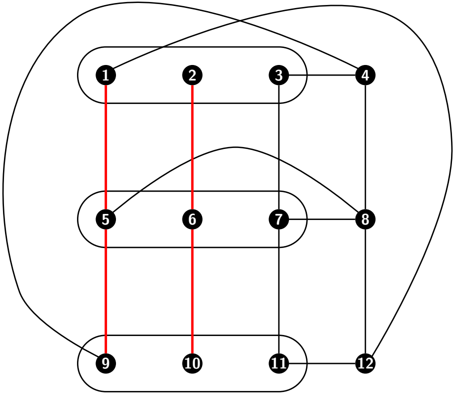

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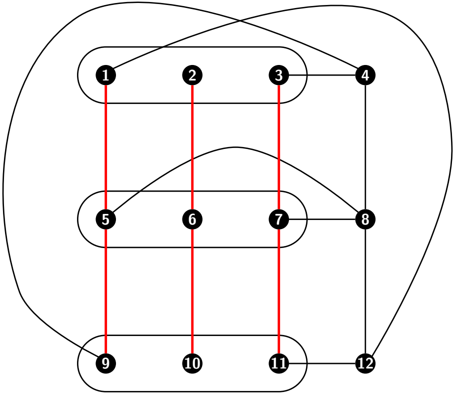

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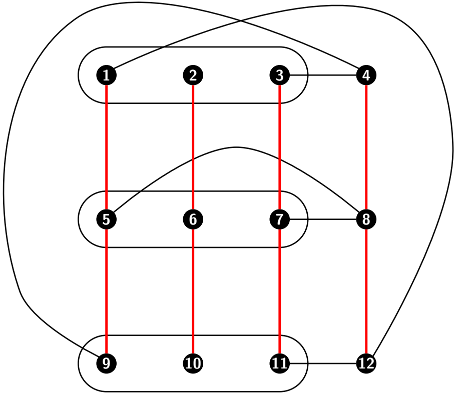

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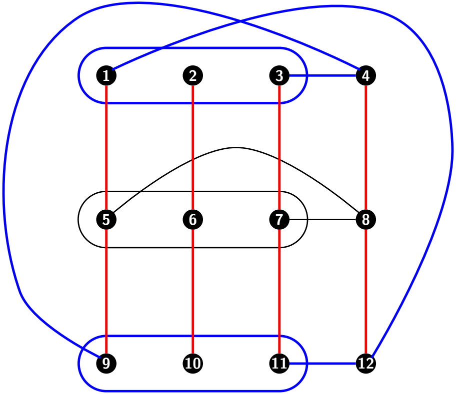

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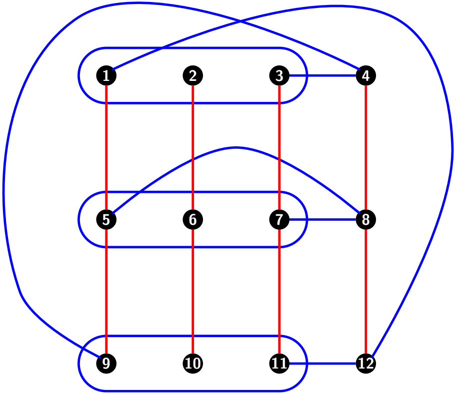
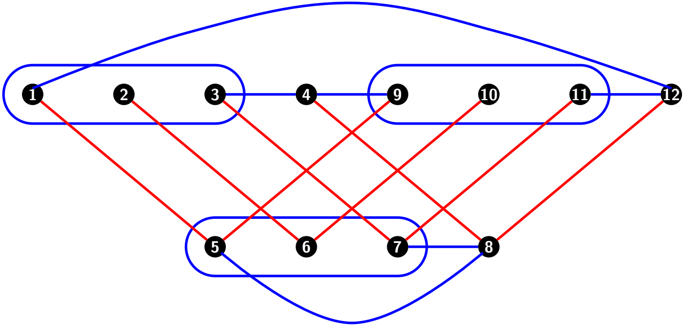


The Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



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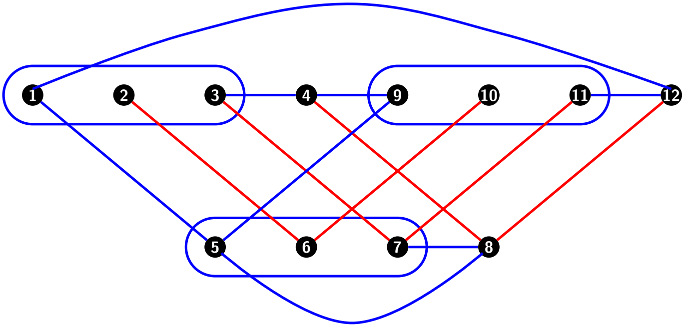

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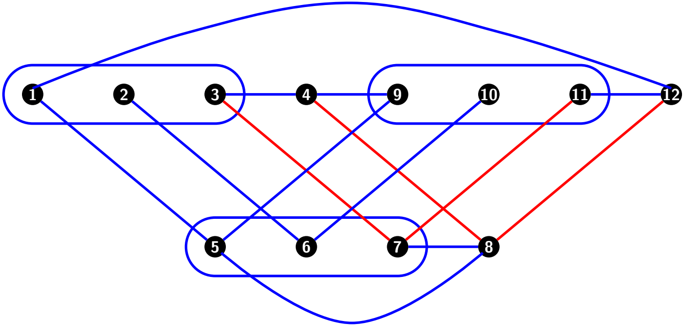

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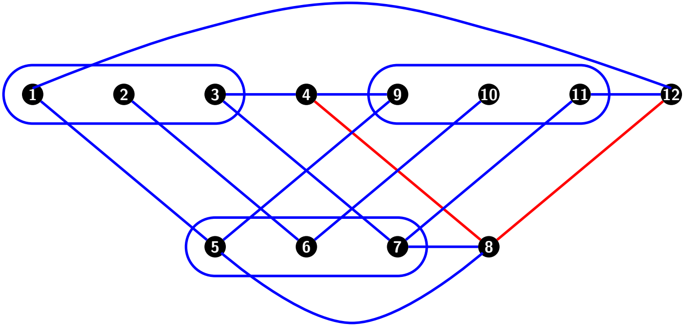

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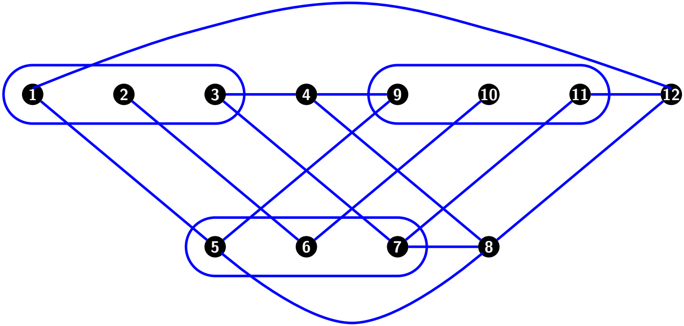

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Every connected Hypergraph has a unique prime factorization.

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Cartesian product of arbitrarily many hypergraphs:

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