Nut graphs with a given automorphism group

Nino Bašić
University of Primorska & IMFM, Slovenia

This is joint work with P. W. Fowler.

Bled, February 15, 2024
39th TBI Winterseminar in Bled

Inštitut za matematiko, fiziko in mehaniko

Spectra of graphs

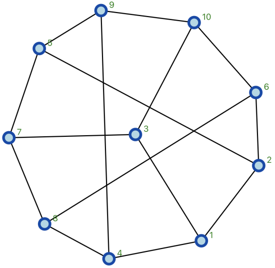
Let G be a finite simple graph of order n with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $E(G)$ denote the edge set of G.

The adjacency matrix of G is the matrix $A(G) = [a_{ij}]_{i,j=1}^{n}$, where

$$a_{ij} = a_{ji} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Example:

$$A(\mathrm{GP}(5,2)) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Spectra of graphs

The spectrum of G, denoted $\sigma(G)$, is the multiset of eigenvalues of $A(G)$.

Examples:

$$\sigma(\mathrm{GP}(5,2)) = \{3, 1^5, (-2)^4\}$$
$$\sigma(K_{3,3}) = \{3, 0^4, -3\}$$

The exponents above give multiplicity of the eigenvalue, e.g. -2 is an eigenvalue of multiplicity 4 in $A(\mathrm{GP}(5,2))$. The multiplicity of the 0 eigenvalue is called nullity and denoted $\eta(G)$.

The eigenvalues are often ordered in non-increasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Singular and core graphs

A graph G is a singular graph if it has a zero eigenvalue.

The graph $K_{3,3}$ is singular, the graph $\mathrm{GP}(5,2)$ is non-singular.

A special class of singular graphs consists of the core graphs, graphs of which the kernel of the adjacency matrix contains a full vector. A full vector is a vector with no zero entry.

For example, the kernel (also called null space) of $A(K_{3,3})$ is

$$\ker A(K_{3,3}) = \operatorname{span}\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

The local condition

An eigenvector $\mathbf{v}$ can be viewed as a weighting of vertices, i.e. a mapping $\mathbf{v}\colon V(G) \to \mathbb{R}$.

A vector $\mathbf{v} \in \ker A$ if and only if for each vertex $v \in V(G)$ the sum of entries over the open neighbourhood $N_G(v)$ equals 0:

$$\sum_{u \in N_G(v)} \mathbf{v}(u) = 0$$

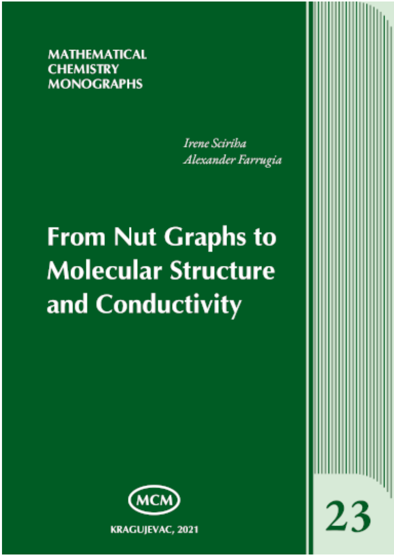
The above equation is called the local condition.

Nut graphs

A simple graph G is a nut graph if G is a core graph with $\eta(G) = 1$.

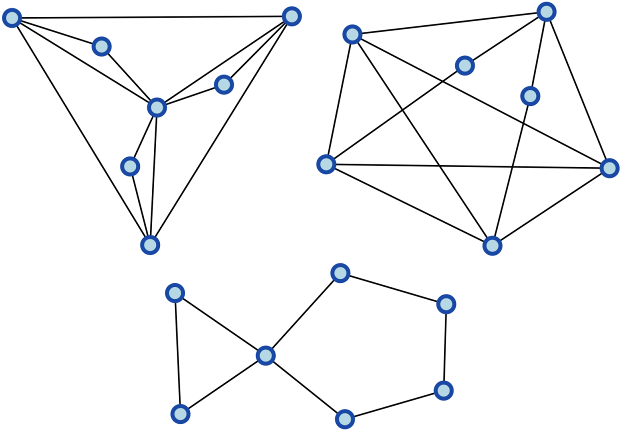
The term nut graph was coined in 1998 by Ivan Gutman and Irene Sciriha.

Why nut graphs?



The smallest nut graphs (aka. Sciriha graphs)

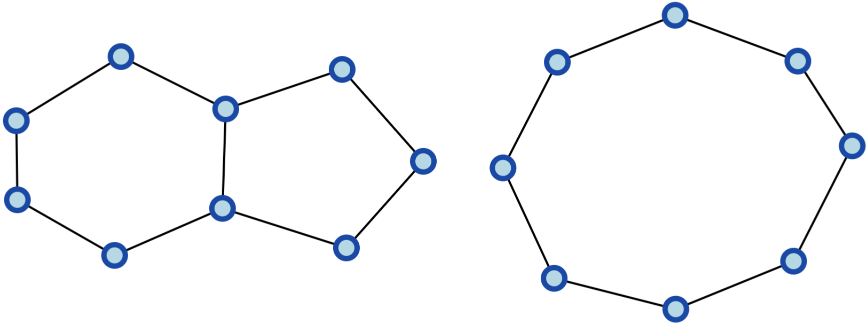
Most authors require that a nut graph has $n \geq 2$ vertices.
K_1 could, in principle, be considered as the ‘trivial’ nut graph.
It is known that a nut graph must have at least 7 vertices.



Pencil-and-paper method

Application of the local condition

Bad examples:



Some properties of nut graphs

Some simple properties of nut graphs:

1. Every nut graph is connected.
2. Every nut graphs is non-bipartite.
3. Every nut graph is leafless (i.e. it have no vertices of degree one).

Number of nut graphs

Order	Nut graphs	Connected graphs	% of nuts
0 – 6	0	143	0.0000
7	3	853	0.0035
8	13	11117	0.0012
9	560	261080	0.0021
10	12551	11716571	0.0011
11	2060490	1006700565	0.0020
12	208147869	164059830476	0.0013
13	96477266994	50335907869219	0.0019

See `https://houseofgraphs.org/meta-directory/nut` for more.

Nut graph can be found in the following graph classes: chemical graphs, cubic graphs, regular graphs, planar graphs, cubic polyhedra (planar 3-connected), fullerenes, . . .

Constructions

Obtaining bigger nuts from smaller nuts

A few such constructions:

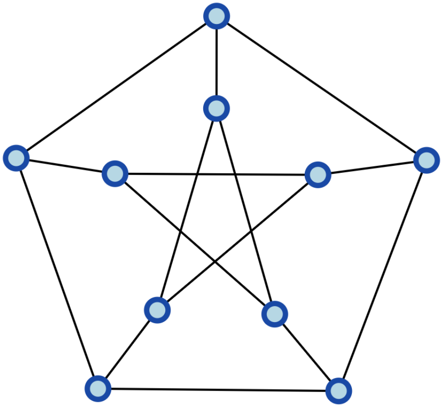
1. the bridge construction
2. the subdivision construction
3. the coalescence construction

Examples?

Symmetry

Informally speaking

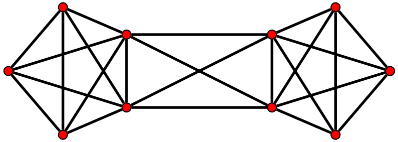
Can we rearrange the vertices somehow and still keep the same graph?



Symmetry

Full automorphism groups of nut graphs on 10 vertices

$\lvert \mathrm{Aut}(G) \rvert$	# graphs
1	8951
2	3101
4	394
6	9
8	58
10	1
12	19
16	5
20	1
24	3
32	3
36	2
48	2
72	1
288	1



Question

D. Kőnig, *Theorie der endlichen und unendlichen Graphen*, 1936.

Abbildung von G in sich darstellt. Auch die „Umkehrung" A_α^{-1} von A_α ist eine Abbildung von G in sich. Somit ist A eine Gruppe, die — als abstrakte Gruppe — die **Gruppe des Graphen** G genannt werden kann. In diesem Zusammenhang kann folgendes Problem (auf welches wir in diesem Buche nicht eingehen) aufgestellt werden. Wann kann eine gegebene abstrakte Gruppe als die Gruppe eines Graphen aufgefaßt werden und — ist dies der Fall — wie kann dann der entsprechende Graph konstruiert werden? Dieselbe Frage läßt sich auch für gerichtete Graphen stellen.

Neben der Gleichheit spielt noch eine zweite Verwandtschaft zwischen

Cubic graphs for a given abstract group

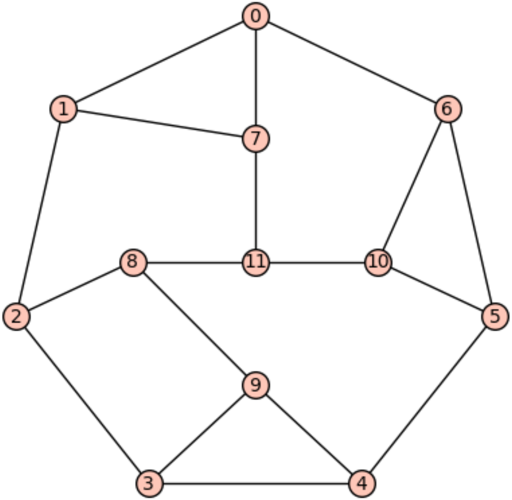
R. Frucht, Graphs of degree three with a given abstract group, *Canadian Journal Mathematics* **1** (1949) 365–378.

Theorem (Frucht, 1949)

For every finite group Γ *there exists a cubic (3-regular) connected graph* G *such that* $\mathrm{Aut}(G) \cong \Gamma$.

Frucht graph

One of the five smallest cubic asymmetric graphs



Multiplier constructions

Theorem

Let G be a connected $(2t)$-regular graph, where $t \geq 1$. Let $\mathcal{M}_3(G)$ be the graph obtained from G by fusing a bouquet of t triangles to every vertex of G. Then $\mathcal{M}_3(G)$ is a nut graph.

Regular graphs for a given abstract group

G. Sabidussi, Graphs with given group and given graph-theoretical properties, *Canadian Journal of Mathematics* **9** (1957) 515–525.

Theorem (Sabidussi, 1957)

For every finite group Γ *and every* $d \geq 3$ *there exists a* d*-regular connected graph* G *such that* $\mathrm{Aut}(G) \cong \Gamma$.

n	id	$\lvert G\rvert$
1	1	10
2	1	9
3	1	14
4	1	64
4	2	64
5	1	80
6	1	96
6	2	96
7	1	112

n	id	$\lvert G\rvert$
8	1	128
8	2	128
8	3	128
8	4	128
8	5	160
9	1	144
9	2	144
10	1	160
10	2	160

Nut graphs for a given abstract group

Theorem

For every finite group Γ there exists a nut graph G such that $\mathrm{Aut}(G) \cong \Gamma$.

Proof.

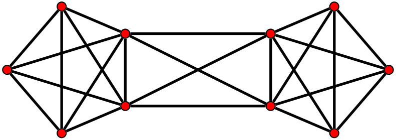
Nut graphs for small groups

n	id	$\lvert\text{Sabi}\rvert$	$\lvert\text{Nut}\rvert$
1	1	10	230
2	1	9	207
3	1	14	322
4	1	64	1472
4	2	64	1472
5	1	80	1840
6	1	96	2208
6	2	96	2208
7	1	112	2576

n	id	$\lvert\text{Sabi}\rvert$	$\lvert\text{Nut}\rvert$
8	1	128	2944
8	2	128	2944
8	3	128	2944
8	4	128	2944
8	5	160	3680
9	1	144	3312
9	2	144	3312
10	1	160	3680
10	2	160	3680

Out nut graphs are slightly large

Recall me?



$|\mathrm{Aut}(G)| = 288$

Sabidussi: 10368

Nut graph: 238464

Regular nut graphs for a given abstract group

Theorem

For every finite group Γ *there exists an* 8*-regular nut graph* G *such that* $\mathrm{Aut}(G) \cong \Gamma$.

Long story short:

For every finite group Γ there exists a nut graph G such that $\mathrm{Aut}(G) \cong \Gamma$.

Thank you!